

Fractal Geometry of Golden Concepts in Fibonacci Sequence

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ABSTRACT: *Fractal geometry is the natural and appropriate geometry to describe unconventional systems, structures and behavior. Fractal structures can describe complex shapes and systems in nature, science and society. In this article we use fractal geometry to describe and analyze the golden concepts associated with the Fibonacci sequence which are the golden mean and the golden angle with their applications in the biological sciences.*

Key Words: *Fractal geometry, Fibonacci sequence, golden mean, golden angle, fractal structure.*

Introduction

Fractal geometry is the appropriate geometry that can be used to describe complex, complicated and unconventional shapes and structures in both natural and artificial systems. To understand life, objects, structures and behavior of systems around us we must appreciate chaos, fractals and turbulence. To explain, analyze and understand fractal and chaotic behavior in both natural and artificial systems we use the mathematics of chaos theory. Chaos theory is a scientific discipline which is focused on the study of nonlinear systems, which are generally complex and unpredictable (Brian, 1993). The cause of unpredictability in nonlinear systems is extreme sensitivity to initial conditions-what is referred to as the butterfly effect. The concept means that with a complex non-linear system, very small changes in the starting conditions of a system will result in dramatically different and large changes in the outputs for that system. Chaos is an unpredictable behavior occurring in a deterministic system (Alligood, Sauern & York, 1996). Chaos which is caused by extreme sensitivity to initial conditions in the parameters of a system explains the complexity, the dual unpredictability and determinism in a lot of physical and social systems. The property of sensitivity to initial conditions can be quantified as:

$$|\delta x(t)| \approx e^{\lambda t} |\delta x(0)| \dots \dots \dots (1)$$

Where λ, t mean rate of separation of trajectories of the system in the fractal diagram. In this article we will illustrate the relationship between fractal geometry and natural systems by considering the fractal properties of the golden mean and the golden angle associated with the Fibonacci sequence.

MATERIALS AND METHODS

Properties of objects with Fractal Features

A fractal is a complicated geometric figure that, unlike a conventional complicated figure, does not simplify when it is magnified. Fractal geometry is used to describe trajectories and structures occurring in nature, science and society especially those produced by chaotic dynamical systems. The term “fractal” was coined in the 1960's by B. Mandelbrot, a mathematician at IBM. The word ‘fractal’ is derived from the Latin word *fractus*, meaning broken or fragmented.

It is generally acknowledged that fractals have some or all of the following properties: complicated structure at a wide range of length scales, repetition of structures at different length scales (self-similarity), and a fractal dimension that is not an integer. Among the geometrically constructed mathematical fractal objects are the Cantor set, the Sierpinski triangle, Koch curve and the fractal cube.

Mathematically, a fractal is a set with a fractional and non-integral dimension greater than its dimension. A fractal can also be described as a function which is continuous, yet non-differentiable and is auto-correlated over a range of scales. Fractals possess the property of self-similarity. Their structure (e.g statistical roughness) looks similar at all scales of observation. Fractals are typically parameterized by their fractal dimension D which is always greater than the Euclidean dimension. Fractals occupy more space than their Euclidean counterparts. The following table gives the fractal dimensions of some basic mathematical fractal structures:

Table 1: Fractal Dimensions of some Fractals

Cantor set	$\ln 2 / \ln 3$
Koch carpet	$\ln 4 / \ln 3$
Sierpinski carpet	$\ln 8 / \ln 3$
Fractal cube	$\ln 6 / \ln 2$
Menger sponge	$\ln 20 / \ln 3$

Results and Discussion

The Fibonacci Sequence

One way to define a sequence is to give a recursive formula or difference equation. The Fibonacci recursive formula or difference equation is given by:

$$F_{n+1} = F_n + F_{n-1}, F_1 = F_2 = 1 \text{ for } (n \geq 2). \dots\dots\dots(2)$$

The difference equation (11) defines a famous sequence whose terms are called the Fibonacci numbers. The first few terms are given are given by:

1, 1, 2, 3, 5, 8, 13, 21, 34. We will solve (12) by the characteristic equation method as follows:

$F^2 - F - 1 = 0$ is the associated characteristic equation which has the roots given by: $F_1 = \frac{1+\sqrt{5}}{2}$ or $F_2 = \frac{1-\sqrt{5}}{2}$.

The general solution is given by $F_n = K_1\left(\frac{1+\sqrt{5}}{2}\right)^n + K_2\left(\frac{1-\sqrt{5}}{2}\right)^n \dots\dots\dots(3)$

Where $K_1 = F_1 / \sqrt{5}$ and $K_2 = -F_2 / \sqrt{5}$ which gives the required solution as:

$$F_n = \frac{1}{\sqrt{5}} \left\{ \left(\frac{1+\sqrt{5}}{2} \right)^{n+1} - \left(\frac{1-\sqrt{5}}{2} \right)^{n+1} \right\} \text{ for } n = 0, 1, 2, \dots\dots\dots(4)$$

The Fibonacci sequence was encountered by Fibonacci also known as Leonardo of Pisa (circa 1175-1250) as a problem involving the number of offspring of rabbits. The Fibonacci sequence models the number of rabbits expected at the end of every month starting with a pair of rabbits which can produce a pair every month which in turn becomes productive after every month. In particular the Fibonacci mathematical series was discovered in 1209 by Leonardo of Pisa, known as Fibonacci while computing the total number of adult rabbits in successive months starting with a single adult rabbit pair and assuming that each adult rabbit pair produces one pair of offspring each month and that baby rabbit pairs became adults in one month time.

Golden Concepts in Fibonacci Sequence

The ratio of adjacent elements of the Fibonacci sequence approaches the irrational number $\tau = (1 + \sqrt{5})/2 \cong 1.618$ in the limit. The number τ , is the solution to the algebraic equation $1+x = x^2$ which implies that $1+\tau = \tau^2$. Thus the double geometric sequence given by:

$$\frac{1}{\tau^3}, \frac{1}{\tau^2}, \frac{1}{\tau}, 1, \tau, \tau^2, \tau^3, \dots\dots\dots(5)$$

is the Fibonacci sequence since it has the property that each term is equal to the sum of the earlier two terms and also the ratio of each term to the earlier term is equal to the **golden mean** τ . It is the only geometric series which is also a Fibonacci sequence. Self-similar structures have in their geometrical design the noble numbers (numbers which are functions of the golden mean) which are all characterized by five-fold symmetry of the pentagon and dodecahedron. For example, the ratio of the length of the diagonal to the side in a regular pentagon is equal to the golden mean. The commonly found shapes in nature are the helix and the dodecahedron which have signatures of self-similarity underlying Fibonacci numbers. The branching network in the summation process of the Fibonacci sequence is a hierarchy of self-similar networks which are fractals.

The association of the **golden mean** and the **golden angle** with growth of self-similar patterns has been established quantitatively in plant phyllotaxis in botany. Phyllotaxis is the study of the arrangement of all plant elements, which originate as primordia on the shoot apex. The botanical elements of a plant include branches, leaves, petals, stamens, sepals, florets, etc. These plant elements begin their existence as primordia in the neighborhood of the undifferentiated shoot apex (extremity). Observations in botany show that in about 92% of plants studied worldwide, primordia emerge as protuberances at locations such that the angle subtended at the apical centre by two successive primordia is equal to the **golden angle** $\varphi = 2\pi(1 - \frac{1}{\tau})$ corresponding to approximately 137.5 degrees.

Summary and Conclusion

Fractals describe- non Euclidean objects generic to nature such as tree roots, tree branches and river basins. The tree structure generated by the process of getting the Fibonacci sequence has been shown

to be a fractal with golden concepts of golden mean and golden angle. The golden concepts were shown to be significant in science and nature, especially in the fractal geometry of botanical structures and mathematical polygonal figures. Fractal behavior can be observed in many natural and artificial systems or models and shall always be a mystery, a paradox, a puzzle, an enigma and a riddle in nature.

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